# A unified dressing method for *C*- and *S*-integrable hierarchies; the particular example of a (3 + 1)-dimensional *n*-wave equation



2004 J. Phys. A: Math. Gen. 37 6557

(http://iopscience.iop.org/0305-4470/37/25/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 6557–6571 PII: S0305-4470(04)77897-X

# A unified dressing method for $C$- and $S$-integrable hierarchies; the particular example of a (3 + 1)-dimensional $n$-wave equation

**Alexandre I Zenchuk**

Centre of Nonlinear Studies of L D Landau Institute for Theoretical Physics (International Institute of Nonlinear Science), Kosygina 2, Moscow 119334, Russia

E-mail: zenchuk@itp.ac.ru

Received 17 March 2004
Published 9 June 2004
Online at stacks.iop.org/JPhysA/37/6557
doi:10.1088/0305-4470/37/25/009

**Abstract**
This paper presents a modification of the dressing method for solving new classes of multidimensional nonlinear partial differential equations (PDE). First, we combine a hierarchy of matrix linearizable ($C$-integrable) equations with a hierarchy of matrix equations integrable by inverse spectral transformation ($S$-integrable equations). The solution manifold of the associated $(n+1)$-dimensional nonlinear PDE has an arbitrary dependence on $n$ variables. This allows us to outline an algorithm for solving known integrable equations with arbitrary forcing terms. Second, we combine two different $S$-integrable hierarchies (two $n$-wave systems in our example). The available solution manifold has an arbitrary dependence on two variables. Multiple-scale expansion is useful in both cases for the purpose of simplification and revealing physical applications of the derived nonlinear PDE. We deal with a (3 + 1)-dimensional space of independent variables, but there is no formal restriction on the dimension.

PACS numbers: 02.30.Jr, 02.30.Ik

## 1. Introduction

Study of nonlinear partial differential equations (PDE) together with their applications in physics represents a significant area of mathematical physics. The interest in it has been enhanced during the last few decades due to the revealing of two large solvable classes of nonlinear PDE. These are equations integrable by inverse spectral transformation (IST) (so-called $S$-integrable equations) [1–3] and linearizable (or $C$-integrable) equations [4–9]. Some of these equations are in turn applicable in different branches of physics, such as hydrodynamics, plasma physics, superconductivity and nonlinear optics. It is well known that different versions of the dressing method are very successful tools for solving nonlinear PDE. The most famous are the Zakharov–Shabat dressing method [10, 11] based on properties of Volterra-type integral operators, the $\bar{\partial}$-problem [12–14] based on properties of Fredholm-type integral operators and the Sato approach [15–18] based on the properties of pseudo-differential operators. Although all these dressing methods have been used only for PDE integrable by IST, it has been shown [19] that there is another type of equations (maybe not integrable by IST) which allow a properly modified dressing procedure for construction of a large manifold of their solutions. But the technique proposed left many questions open. For instance, it is not clear whether derived nonlinear PDE can be linearized by some substitution. Also it was difficult to characterize the manifold of available solutions.

In this paper we replace the algebraic operator with an integral one, generalize the system of equations introducing a set of additional parameters (independent variables of nonlinear PDE) and significantly modify the algorithm given in [19]. This allows us to simplify the description of PDE properties and give more information about the solution manifold as well as relations for the classical solvable PDE (both $S$- and $C$-integrable).

Thus, the basic object is the following $N \times N$ matrix integral equation:

$$\Phi + \chi \equiv \Phi(\lambda, \mu; t) + \chi(\lambda, \mu; t) = \int_{D_\nu} \Psi(\lambda, \nu; t) U(\nu, \mu; t)\,\mathrm{d}\nu \equiv \Psi * U \tag{1}$$

where $\lambda = (\lambda_1, \dots, \lambda_{N_\lambda})$, $\mu = (\mu_1, \dots, \mu_{N_\mu})$ and $\nu = (\nu_1, \dots, \nu_{N_\nu})$ are vector spectral parameters with different lengths in general; $t = (t_1, \dots, t_{\dim(t)})$ is a set of independent variables of the nonlinear PDE, $\dim(t)$ may be either finite or infinite; integration is over the whole space $D_\nu$ of the appropriate spectral parameter; $\Phi$, $\chi$, $\Psi$ and $U$ are $N \times N$ matrix functions of arguments. Star means integration over the space of the 'inner' spectral parameter: $f * g \equiv \int_{D_\nu} f(\lambda_1, \dots, \nu) g(\nu, \dots, \mu_n)\,\mathrm{d}\nu$. We require $\Psi$ to be an invertible operator; i.e. equation (1) can be solved uniquely for $U$. By definition, operator $\mathcal{A}(\lambda, \mu)$ is invertible if there are operators $\mathcal{A}_L^{-1}(\lambda, \mu)$ and $\mathcal{A}_R^{-1}(\lambda, \mu)$ such that $\int_{D_\nu} \mathcal{A}(\lambda, \nu)\mathcal{A}_R^{-1}(\nu, \mu)\,\mathrm{d}\nu = \int_{D_\nu} \mathcal{A}_L^{-1}(\lambda, \nu)\mathcal{A}(\nu, \mu)\,\mathrm{d}\nu = \delta(\lambda - \mu)$. The functions $\Phi$ and $\Psi$ are related by means of the *compatible* system of linear integral–differential equations which, however, introduce the set of variables $t$:

$$M_i * \Psi = \sum_k L_{ik} * \Phi * C_{ki} \qquad C_{ki} = C_{ki}(\lambda, \mu; t) \qquad i = 1, 2, \dots \tag{2}$$

where $M_j = M_j(\lambda, \mu; \partial_{t_1}, \partial_{t_2}, \dots)$ are first order and $L_{jk} = L_{jk}(\lambda, \mu; \partial_{t_1}, \partial_{t_2}, \dots)$ are arbitrary order linear differential operators with matrix coefficients depending on $\lambda$ and $\mu$. This overdetermined system together with its compatibility condition defines $\Psi$ and $\Phi$. Finally, the same compatibility condition, with $\Phi$ defined by equation (1), results in nonlinear PDE whose solution is expressed in terms of $U$.

With this preliminary discussion complete, we now derive some general equations using the following simplified version of the system (2):

$$\Psi_{t_i} = S_i * \Phi * C_i \qquad i = 1, 2, \dots \tag{3}$$

where $S_i(\lambda, \mu; t)$ and $C_i(\lambda, \mu; t)$ are known functions of $t$, as will be seen later. The compatibility condition for the system (3) has the form

$$\begin{aligned} &S_{i t_j} * \Phi * C_i - S_{j t_i} * \Phi * C_j + S_i * \Phi * C_{i t_j} - S_j * \Phi * C_{j t_i} \\ &\qquad + S_i * \Phi_{t_j} * C_i - S_j * \Phi_{t_i} * C_j = 0 \end{aligned} \tag{4}$$

which is a linear system of compatible integral–differential equations for the function $\Phi$. Solving this equation, substituting the result in equation (3) and integrating it, we obtain $\Psi$:

$$\Psi(\lambda, \mu; t) = \partial_{t_i}^{-1}(S_i * \Phi * C_i)(\lambda, \mu; t) + E(\lambda, \mu) + F_i(\lambda, \mu; t_2, t_3, \dots). \tag{5}$$

Here $E$ is invertible operator, the functions $F_i$ provide compatibility of the system (5). Being invertible (owing to $E$), the operator $\Psi$ provides the unique solution to equation (1).

However, equation (4) may be given in another form after substitution of equation (1) for $\Phi$ and equation (3) for $\Psi_{t_i}$:

$$\begin{aligned} S_{i\,t_j} * (\Psi * U - \chi) * C_i - S_{j\,t_i} * (\Psi * U - \chi) * C_j + S_i * (\Psi * U - \chi) * C_{i\,t_j} \\ - S_j * (\Psi * U - \chi) * C_{j\,t_i} + S_i * \big(S_j * (\Psi * U - \chi) * C_j * U \\ + \Psi * U_{t_j} - \chi_{t_j}\big) * C_i - S_j * \big(S_i * (\Psi * U - \chi) * C_i * U \\ + \Psi * U_{t_i} - \chi_{t_i}\big) * C_j = 0 \end{aligned} \tag{6}$$

which is a nonlocal equation quadratic in $U$. This may result in nonlinear PDE for fields expressed in terms of $U$, $S_i$ and $C_i$. To provide for this possibility we must impose specific dependences of the functions $S_i$ and $C_i$ on their arguments. Thus, hereafter we use the following relations:

$$S_i(\lambda, \mu; t) = \delta(\lambda - \mu) \tag{7}$$

$$C_i(\lambda, \mu; t) = \int_{D_\nu} A_i(\lambda, \nu) p_1(\nu; t)\, \mathrm{d}\nu\, p_2(\mu) + c_1(\lambda) B_i c_2(\mu; t) \equiv A_i * p_1(t) p_2 + c_1 B_i c_2(t) \tag{8}$$

where $A$ is an invertible operator, $A_i * A_j = A_j * A_i$, $[B_i, B_j] = 0$ and $[*, *]$ means the commutator of two matrices. Then equation (4) is equivalent to the following set of three integral–differential equations for $\Phi$ and $c_2$:

$$(\Phi * A_1 * p_1(t))_{t_j} - (\Phi * A_j * p_1(t))_{t_1} = 0 \tag{9}$$

$$(\Phi * c_1)_{t_j} B_1 - (\Phi c_1)_{t_1} B_j = 0 \tag{10}$$

$$B_1 c_{2t_j} - B_j c_{2t_1} = 0 \tag{11}$$

while equation (3) reads

$$\Psi_{t_i} = \Phi * C_i. \tag{12}$$

Equations (9) and (10) define the functions $\Phi$ and have to be compatible. The simple way to provide this compatibility is by 'splitting' these two equations in the following way:

$$\Phi_{t_j} * A_1 - \Phi_{t_1} * A_j = 0 \tag{13}$$

$$A_1 * \partial_{t_j} p_1 - A_j * \partial_{t_1} p_1 = 0 \tag{14}$$

$$A_i * c_1 = c_1 B_i. \tag{15}$$

From equations (1), (12), (13) one derives the following nonlinear equation instead of (6):

$$\begin{aligned} \Psi * \big(U_{t_j} * A_1 + U * C_j * U * A_1 - U_{t_1} * A_j - U * C_1 * U * A_j\big) \\ + \chi_{t_1} * A_j - \chi_{t_j} * A_1 + \chi * (C_1 * U * A_j - C_j * U * A_1) = 0. \end{aligned} \tag{16}$$

Although the equations derived in [19] may be solved with our algorithm, they will not be discussed in this paper. Here we consider two other examples of multidimensional systems. The first of them (section 2) represents a combination of $C$- and $S$-integrable ($n$-wave) equations. This is a $(3 + 1)$-dimensional system, having solutions with an arbitrary dependence on *three variables* at most. Among these systems are known integrable models with arbitrary forcing terms. The second example (section 3) combines two *S*-integrable equations (two different matrix *n*-wave equations) [20, 21]. Properly introduced multiple-scale expansion of this system results in a (3 + 1)-dimensional *n*-wave equation. Its solutions may have an arbitrary dependence on *two variables* at most. Some reductions of this system are given. Both cases have extensions into higher dimensions.

We emphasize that we consider the problem of construction the families of solutions to the derived nonlinear equations regardless of the completeness of their integrability.

## 2. The generalized hierarchy of linearizable and integrable by IST (*n*-wave) systems

In this section $\chi = 0$, $A_j = \underbrace{A * \cdots * A}_{j} \equiv A^j$, $B_j = B^j$, where $A(\lambda, \mu)$ is an invertible operator and $B$ is a nondegenerate constant matrix. Applying the operator $\Psi^{-1}$ to equation (16) from the left, one obtains

$$E_j = U_{t_j} * A + U * A^j * p_1 p_2 * U * A + U * c_1 B^j c_2 * U * A - \left(U_{t_1} * A^j + U * A * p_1 p_2 * U * A^j + U * c_1 B c_2 * U * A^j\right) = 0 \tag{17}$$

where $\partial \equiv \partial_{t_1}$. We may derive a nonlinear system for the functions

$$u = p_2 * U * c_1 \qquad q_n = p_2 * U * A^n * p_1 \qquad v_n = \partial^n c_2 * U * c_1 \qquad w_{mn} = \partial^m c_2 * U * A^n * p_1 \tag{18}$$

which has the following form:

$$\mathcal{E}_{u;j} = p_2 * E_j * c_1 = u_{t_j} - u_{t_1} B^{j-1} + q_j u - q_1 u B^{j-1} + u B^j v_0 - u B v_0 B^{j-1} = 0 \tag{19}$$

$$\mathcal{E}_{q;jn} = p_2 * E_j * A^n * p_1 = q_{n t_j} - q_{n+j-1_{t_1}} + q_j q_n - q_1 q_{n+j-1} + u B^j w_{0n} - u B w_{0(n+j-1)} = 0 \tag{20}$$

$$\mathcal{E}_{v;jn} = \partial^n c_2 * E_j * c_1 = v_{n t_j} - v_{n t_1} B^{j-1} + [v_{n+1}, B^{j-1}] + w_{nj} u - w_{n1} u B^{j-1} + v_n B[B^{j-1}, v_0] = 0 \tag{21}$$

$$\mathcal{E}_{w;jmn} = \partial^m c_2 * E_j * A^n * p_1 = w_{mn t_j} - w_{m(n+j-1)_{t_1}} - B^{j-1} w_{(m+1)n} + w_{(m+1)(n+j-1)} + w_{m+j} q_n - w_{m1} q_{n+j-1} + v_m B^j w_{0n} - v_m B w_{0(n+j-1)} = 0. \tag{22}$$

The number of fields in this system can be reduced (we rearrange equations in an order that emphasizes the relation to the classical systems):

$$\begin{gathered} \mathcal{E}_{q;jn} (j = 2, n = 0, \ldots, 3 \text{ and } j = 3, n = 0) \qquad \mathcal{E}_{v;30} - B\mathcal{E}_{v;20} - \mathcal{E}_{v;20} B \quad \mathcal{E}_{u;2} \\ \mathcal{E}_{w;400} - \mathcal{E}_{w;202} - B\mathcal{E}_{w;201} - B^2 \mathcal{E}_{w;200} (n = 0, \ldots, 3) \\ \mathcal{E}_{w;30n} - \mathcal{E}_{w;20(n+1)} - B\mathcal{E}_{w;20n} (n = 0, \ldots, 3) \end{gathered} \tag{23}$$

or, in explicit form ($w_n \equiv w_{0n}$, $v \equiv v_0$),

$$q_{n t_j} - q_{n+j-1_{t_1}} + q_j q_n - q_1 q_{n+j-1} + u B^j w_n - u B w_{n+j-1} = 0 \qquad j = 2 \quad n = 0, \ldots, 3 \quad \text{and} \quad j = 3 \quad n = 0 \tag{24}$$

$$v_{t_3} - B v_{t_2} - v_{t_2} B + B v_{t_1} B - [v, B]B[v, B] - B w_2 u - w_2 u B + w_3 u + B w_1 u B = 0 \tag{25}$$

$$u_{t_2} - u_{t_1} B + q_2 u - q_1 u B + u B[B, v] = 0 \tag{26}$$

$$\left(\partial_{t_4} - B^2 \partial_{t_2}\right) w_0 - \left(\partial_{t_2} - B\partial_{t_1}\right)(w_2 + B w_1) + (B w_1 - w_2) q_2 + B(B w_1 - w_2) q_1 + (w_4 - B^2 w_2) q_0 + [B, v] B w_2 + B[B, v] B w_1 - [B^2, v] B^2 w_0 = 0 \tag{27}$$

$$\left(\partial_{t_3} - B\partial_{t_2}\right)w_n - \left(\partial_{t_2} - B\partial_{t_1}\right)w_{n+1} + (w_3 - Bw_2)q_n - (w_2 - Bw_1)q_{n+1}$$
$$+[B, v]B(w_{n+1} - Bw_n) = 0 \qquad n = 0, \ldots, 3. \tag{28}$$

Thus, this system is (3 + 1)-dimensional. We see that this is a subsystem of a more general system (19)–(22). The important fact for the integrability of equations (24) and (28) is that missed equations do not impose any constraint since they are commuting flows for appropriate equations of the system (24)–(28). For instance, the first equation (19) gives us the evolution of $u$ with respect to $t_j$, $j \geqslant 2$. Direct calculation shows that $u_{t_2 t_j} = u_{t_j t_2}$. Thus we pick out just equation (19) with $j = 2$; see equation (26).

Consider several reductions of the system (24)–(28). We start with those reductions which lead to classical integrable systems.

*Reduction 1. A* $(2+1)$*-dimensional matrix n-wave equation.* Let expression (8) for $C_i$ have only the second term; thus $w_n = q_n = u = 0$ and the following reduction is possible in equation (25): $v_{\beta\alpha} = -\bar{v}_{\alpha\beta}, \beta > \alpha, t_j \to it_j$. This equation becomes a (2 + 1)-dimensional $n$-wave equation for off-diagonal elements of matrix $v$. Other equations disappear.

*Reduction 2. A* $(2+1)$*-dimensional matrix C-integrable system.* Let expression (8) for $C_i$ have only the first term. Thus $w_n = v = u = 0$ and we stay with equation (24) for $q_n$. One can show that $q_n$ can be expressed in terms of solutions $f$ of the linear PDE $f_{t_2 t_2} = f_{t_1 t_3}$ by matrix Hopf substitution [7]. In fact, in this case $\Psi(\lambda, \mu; t) = \partial_{t_j}^{-1}(\Phi * A * p_1(t))(\lambda) p_2(\mu) + \delta(\lambda - \mu)$ and equation (1) becomes

$$\Phi = \partial_{t_j}^{-1}(\Phi * A^j * p_1(t)) p_2 * U + U.$$

Applying operators $*A^j * p_1(t)$ and $p_2 *$ to these equations from the right and from the left respectively, one gets

$$q_j = p_2 * U * A^j * p_1(t) = \left(I + \partial_{t_j}^{-1}(p_2 * \Phi * A^j * p_1(t))\right)^{-1} p_2 * \Phi * A^j * p_1(t).$$

Let $f = I + \partial_{t_j}^{-1}(p_2 * \Phi * A^j * p_1(t))$. Then the above linear equation for $f$ may be derived from (13) and (14). Let us change the independent variables:

$$\partial_{t_1} = \partial_t + \partial_y \quad \partial_{t_3} = \partial_t - \partial_y \quad \partial_{t_2} = \partial_x \quad \partial_{t_4} = \partial_z. \tag{29}$$

Then the equation for $f$ becomes a linear wave equation with two space coordinates.

*Reduction 3. An 'almost' C-integrable system.* We illustrate the simple particular approximation of the above system corresponding to an 'almost' linearizable system with a small perturbation coming from the $S$-integrable part. This happens if the second term in equation (8) is small and $B$ is proportional to the small parameter $\varepsilon$: $c_1 = \varepsilon\tilde{c}_1, c_2 = \varepsilon\tilde{c}_2$, $p_i \sim 1$, $B = \varepsilon\tilde{B}, \varepsilon \ll 1$. Then $q_n \sim 1$, $u \sim w_{nm} \sim \varepsilon, v_n \sim \varepsilon^2$. We expand the above system in powers of $\varepsilon$, keeping the term of order $\varepsilon^3$ in equation (24) and leading terms in other equations. Equation (25) can be disregarded for this case. Thus one has

$$q_{n t_j} - q_{n+j-1_{t_1}} + q_j q_n - q_1 q_{n+j-1} - \varepsilon^3 u\tilde{B} w_{n+j-1} = 0$$
$$j = 2 \quad n = 0, \ldots, 3 \quad \text{and} \quad j = 3 \quad n = 0 \tag{30}$$

$$u_{t_2} + q_2 u = 0 \tag{31}$$

$$\partial_{t_4} w_0 - \partial_{t_2} w_2 + w_4 q_0 - w_2 q_2 = 0 \tag{32}$$

$$\partial_{t_3} w_n - \partial_{t_2} w_{n+1} + w_3 q_n - w_2 q_{n+1} = 0 \qquad n = 0, \ldots, 3. \tag{33}$$

The nonlinearizable perturbation is related to the last term in equation (30). In terms of the independent variables $x, y, t$ (29), this system may be transformed to the following one:

$$q_{n\,tt} = q_{n\,xx} + q_{n\,yy} + [q_2 q_n - q_1 q_{n+1}]_x + (\partial_t + \partial_y)[q_2 q_{n+1} - q_3 q_n]$$
$$- \varepsilon^3 (u\tilde{B} w_{n+1})_x \qquad n = 0, 1, 2 \tag{34}$$

$$q_{2_x} - q_{3_t} - q_{3_y} + q_2 q_2 - q_1 q_3 - \varepsilon^3 u\tilde{B} w_3 = 0 \tag{35}$$

$$u_x + q_2 u = 0 \tag{36}$$

$$\partial_z w_0 - \partial_x w_2 + w_4 q_0 - w_2 q_2 = 0 \tag{37}$$

$$(\partial_t - \partial_y) w_n - \partial_x w_{n+1} + w_3 q_n - w_2 q_{n+1} = 0 \qquad n = 0, \dots, 3. \tag{38}$$

The first equation is a variant of the nonlinear wave equation with two space variables and nonlinearity depending on some other fields, satisfying equations (35) and (38). The third space variable $z$ appears only due to these additional fields; see equation (37). If $\varepsilon = 0$, then the system is linearizable by Hopf substitution (see *reduction 2*).

In a similar way one can introduce an 'almost' $S$-integrable system. We will not do that in this paper.

*Reduction 4. A nonlinear n-wave equation with an arbitrary forcing term.* This example is based on equation (25):

$$v_{t_3} - B v_{t_2} - v_{t_2} B + B v_{t_1} B - [v, B] B [v, B] = F(t) \tag{39}$$

$$F(t) = B w_2 u + w_2 u B - w_3 u - B w_1 u B. \tag{40}$$

The off-diagonal part of this equation is a $(2+1)$-dimensional $n$-wave equation with the right-hand side expressed in terms of the functions $u_j$ and $w_j$. It is very important that $w_j$ has an arbitrary dependence on three variables (see the next subsection) depending on an arbitrary function (say, $f$) of three variables. Although this dependence is implicit, we denote this fact by the formula $w_j(t) = w_j(t, f(t_1, t_2, t_3))$ ($f$ is an arbitrary function of its arguments) to emphasize explicitly that the function $f$ is the only one for all $w_j$ (or more exactly: only one of the functions $w_j$ may be an arbitrary function of three variables). Since equation (39) is three-dimensional, this means that formally for any particular forcing term $F(t)$ one can find a function $f$ (and consequently $w_j$) solving equation (40). In this case we are not concerned about other equations of the system (24)–(28). In particular, $F(t)$ may be a nonlinear function of $v$ and (perhaps) its derivatives: $F(t) = H\big(v, v_{t_1}, v_{t_2}, \dots, v_{t_i t_j}, \dots, v_{\eta_1}, v_{\eta_2}, \dots\big)$, where $\eta = (\eta_1, \dots, \eta_{\dim(\eta)})$ is a list of additional variables. Of course this significantly complicates the algorithm for solution construction. The dimension of this type of nonlinear PDE is defined by the dimension of the appropriate $S$-integrable system as well as by the additional dimension $\dim(\eta)$ introduced with the function $H$.

We will mention equation (40) in the next subsection, where the solution manifold for the equations (24) and (28) is discussed. Here we should say that solutions to equation (39) may be found only numerically and details are not given in this paper.

*Reduction 5. A linear wave equation with an arbitrary forcing term.* Like in the previous case, one could take equation (24) and consider a $C$-integrable equation with an arbitrary forcing term. But linear equations with an arbitrary forcing term are more applicable in physics. It is interesting that equation (24) can be approximated by a linear equation with the forcing expressed in terms of $w_j$ if the small parameter $\varepsilon$ is properly introduced. Let $p_i \sim \varepsilon$, $c_i \sim 1$. Then $q_j \sim \varepsilon^2$, $u \sim w \sim \varepsilon$ and, in leading order ($\varepsilon^2$), equation (24) has the form

$$q_{n_{t_j}} - q_{n+j-1_{t_1}} = F_{n,j}(t) \tag{41}$$

$$F_{n,j}(t) = uBw_{n+j-1} - uB^j w_n. \tag{42}$$

In terms of the variables $x, y, t$ (29) the equation for $q = q_0$ has the following form:

$$q_{tt} - q_{xx} - q_{yy} = \tilde{F}(\mathbf{x}) \qquad \mathbf{x} = (x, y, t) \tag{43}$$

$$\tilde{F}(\mathbf{x}) = (\partial_t + \partial_y)(uB^3 w_0 - uB^2 w_1) + \partial_x(uB^2 w_0 - uBw_1). \tag{44}$$

To find $w_j(t, f(t))$ for given $\tilde{F}$ one has to solve two-dimensional (since derivatives $\partial_t$ and $\partial_y$ appear only in linear combination) PDE (44). This is a rather complicated procedure. But if we take $B \sim \varepsilon$ and $\partial_{t_j} \sim \varepsilon$, then the leading part of equation (41) will be of order $\varepsilon^3$ and the expression for $\tilde{F}$ will be simplified:

$$\tilde{F}(\mathbf{x}) = -(uBw_1)_x; \tag{45}$$

i.e. one arrives at the first-order ordinary differential equation for $w_1$. Similarly to in the previous reduction, one can take $\tilde{F}$ as a nonlinear function of $q$ and (perhaps) its derivatives: $\tilde{F}(\mathbf{x}) = H(q, q_{\mathbf{x}}, \ldots, q_{\eta_1}, q_{\eta_2}, \ldots)$, where $\eta$ is a list of additional variables (see *reduction 4*). The dimension of PDE depends on both the dimension of the appropriate linear operator and the additional dimension $\dim(\eta)$ introduced by the function $H$. Equations (44) and (45) will be mentioned in the next subsection, but the complete procedure of solution construction is beyond the scope of this paper.

### 2.1. Construction of solutions

At the beginning of this subsection we show how rich the solution manifold is. Let us represent equation (1) in the following form:

$$U = \Phi - \partial_{t_1}^{-1}[\Phi * A * p_1(t)]p_2 * U - \partial_{t_1}^{-1}[\Phi * c_1 c_2(t)] * U \tag{46}$$

where we substitute the expression for $\Psi$ following from equation (12):

$$\Psi(\lambda, \mu; t) = \left(\partial_{x_1}^{-1}\Phi * C_1\right)(\lambda, \mu; t) + \delta(\lambda - \mu). \tag{47}$$

$c_2(t)$ is defined by equation (11) and depends on an arbitrary function of a single independent variable, which is reflected by the following formula:

$$c_2(\mu; t) = \int_{\Omega_k} \mathrm{e}^{k\sum_i B^i t_i} c_{20}(\mu; k)\,\mathrm{d}k \tag{48}$$

where $k$ is some (complex) parameter. The function $\Phi * c_1$ is defined by equation (10) and has an arbitrary dependence on a single independent variable as well. The function $\Phi * A * p_1$ is defined by equation (9) and consequently has an arbitrary dependence on two variables. In fact, let us write out several equations with variables $t_1, t_2, t_3$:

$$(\Phi * A * p_1)_{t_2} = (\Phi * A^2 * p_1)_{t_1} \qquad (\Phi * A^2 * p_1)_{t_2} = (\Phi * A^3 * p_1)_{t_1}$$
$$(\Phi * A * p_1)_{t_3} = (\Phi * A^3 * p_1)_{t_1}. \tag{49}$$

This is a three-dimensional complete system of three equations. The linear equation for $\varphi = \Phi * A * p_1$ reads $\varphi_{t_2 t_2} = \varphi_{t_1 t_3}$ which has the solution $\varphi(\lambda; t) = \int_{D_k} \varphi_0(\lambda; k_1, k_2)\,\mathrm{e}^{\mathrm{i}\sum_{j=1}^3 k_j t_j}\delta\left(k_2^2 - k_1 k_3\right)\mathrm{d}k$, $k = (k_1, k_2, k_3)$; i.e. the solution has an arbitrary function $\varphi_0(\lambda; k_1, k_2)$ of two variables $k_1$ and $k_2$. Integration is over the three-dimensional space of the real vector parameter $k$.

The formulae for functions $u, v, q_j$ and $w_j$ have the following terms and factors:

$$h_{11} = p_2 * \Phi(t) * c_1 \qquad h_{12} = p_2 * \partial_{t_1}^{-1}[\Phi(t) * c_1 c_2(t)] \tag{50}$$

$$h_{21} = c_2 * \Phi(t) * c_1 \qquad h_{22}^j = p_2 * \Phi(t) * A^j * p_1(t) \tag{51}$$

$$\begin{aligned} &h_{23} = c_2 * \partial_{t_1}^{-1}[\Phi(t) * c_1 c_2(t)] \qquad h_{24} = p_2 * \partial_{t_1}^{-1}[\Phi(t) * A * p_1(t)] \\ &h_{31}^j = c_2 * \Phi(t) * A^j * p_1(t) \qquad h_{32} = c_2 * \partial_{t_1}^{-1}[\Phi(t) * A * p_1(t)]. \end{aligned} \tag{52}$$

Thus we need to realize the functional freedom of these expressions. For this purpose we prove a more general statement; namely, we show explicitly that if $f(\lambda; t)$ and $g(\lambda; t)$ have an arbitrary dependence on $n_1$ and $n_2$ independent variables $t_i$ respectively, then $h(t) = f(t) * g(t)$ may have an arbitrary dependence on at most $n_1 + n_2$ variables $t_i$. This can be easily done for functions having, for instance, Fourier transformations. Let $f(\lambda; t)$ be an arbitrary function of $n_1$ parameters $t_j$, $j = 1, \ldots, n_1$ and $g$ be an arbitrary function of $n_2 \geqslant n_1$ parameters $t_j$, $j = n_1 + 1, \ldots, n_1 + n_2$, where $\lambda$ is a real vector spectral parameter whose length is $n_1$. Let $f$ and $g$ be taken in the form

$$\begin{aligned} f(\lambda; t) &= \int_{\Omega_k} f_0(\tilde{k}) \left( \prod_{m=1}^{n_1} \delta(\lambda - k_m) \right) \exp\left( \mathrm{i} \sum_{j=1}^{\dim(t)} k_j t_j \right) \left( \prod_{m=n_1+1}^{\dim(t)} \delta\big(D_m^f(k)\big) \right) \mathrm{d}k \\ \tilde{k} &= \big(k_1, \ldots, k_{n_1}\big) \end{aligned} \tag{53}$$

$$\begin{aligned} g(\lambda; t) &= \int_{\Omega_q} \exp\left( \mathrm{i} \sum_{j=1}^{\dim(t)} q_j t_j \right) \left( \prod_{m=1}^{n_1} \delta\big(D_m^g(q)\big) \right) \left( \prod_{m=n_1+n_2+1}^{\dim(t)} \delta\big(D_m^g(q)\big) \right) g_0(\lambda; \tilde{q})\, \mathrm{d}q \\ \tilde{q} &= \big(q_{n_1+1}, \ldots, q_{n_1+n_2}\big). \end{aligned} \tag{54}$$

Here $\Omega_k$ and $\Omega_q$ are $\dim(k) = \dim(t)$-dimensional planes of real vector parameters $k$ and $q$ respectively; $D_n^f$ and $D_n^g$ are dispersion relations such that

$$\det \left| \frac{\partial\big(D_{n_1+1}^f \cdots D_{\dim(t)}^f\big)}{\partial\big(k_{n_1+1} \cdots k_{\dim(t)}\big)} \right| \neq 0 \qquad \det \left| \frac{\partial\big(D_1^g \cdots D_{n_1}^g D_{n_1+n_2+1}^g \cdots D_{\dim(t)}^g\big)}{\partial\big(k_1 \cdots k_{n_1} k_{n_1+n_2+1} \cdots k_{\dim(t)}\big)} \right| \neq 0.$$

Then one can carry out integration over $\lambda$ inside the expression $h = f * g$:

$$\begin{aligned} h(t) = \int_{\Omega_k} \int_{\Omega_q} & f_0(\tilde{k}) \exp\left( \mathrm{i} \sum_{j=1}^{\dim(t)} (k_j + q_j) t_j \right) \left( \prod_{m=n_1+1}^{\dim(t)} \delta\big(D_m^f(k)\big) \right) \left( \prod_{m=1}^{n_1} \delta\big(D_m^g(q)\big) \right) \\ & \times \left( \prod_{m=n_1+n_2+1}^{\dim(t)} \delta\big(D_m^g(q)\big) \right) g(\tilde{k}; \tilde{q})\, \mathrm{d}k\, \mathrm{d}q \\ & \tilde{k} = \big(k_1, \ldots, k_{n_1}\big) \quad \tilde{q} = \big(q_{n_1+1}, \ldots, q_{n_1+n_2}\big) \end{aligned} \tag{55}$$

i.e. the function $h(t)$

$$h(t) = \int_{\Omega_{\tilde{k}}} \int_{\Omega_{\tilde{q}}} f_0(\tilde{k}) g(\tilde{k}, \tilde{q})\, \mathrm{e}^{\mathrm{i}\Omega(\tilde{k}, \tilde{q}, t)}\, \mathrm{d}\tilde{k}\, \mathrm{d}\tilde{q}$$

has an arbitrary dependence on $n_1 + n_2$ variables.

In our case $c_2$ is taken for $f$ and $c_{20}(\mu; k) = \delta(\mu - k)$. We see that the function $u$ may have an arbitrary dependence on a single variable (due to the terms $h_{11}$ and $h_{12}$) and two variables (due to $h_{24}$); $v$ may have an arbitrary dependence on two variables (due to the terms $h_{21}$ and $h_{23}$) and three variables (due to $h_{24}$); $q_j$ may have an arbitrary dependence on a single variable (due to $h_{12}$), two variables (due to the term $h^j_{22}$) and three variables (due to $h_{24}$); finally, $w_j$ may have an arbitrary dependence on two variables (due to $h_{23}$) and three variables (due to $h^j_{31}$ and $h_{32}$).

Now we outline the algorithm for construction of solutions to the system (24)–(28). First, one needs to solve the system (13)–(15) for the functions $c_1$, $\Phi$, $p_1$ and $c_2$:

$$\Phi(\lambda, \mu; t) = \int_{\Omega_k} \int_{D_\nu} \Phi_0(\lambda, \nu; k)\, \mathrm{e}^{\eta_1(\nu;k,t)} \phi_0(\nu, \mu; k)\, \mathrm{d}k\, \mathrm{d}\nu \tag{56}$$

$$p_1(\lambda; t) = \int_{\Omega_k} \int_{D_\nu} p_0(\lambda, \nu; k)\, \mathrm{e}^{\eta_2(\nu;k,t)} p_{10}(\nu; k)\, \mathrm{d}k\, \mathrm{d}\nu \tag{57}$$

$$A * c_1 = c_1 B \tag{58}$$

where $\eta_i(\mu; k, t) = \sum_{j=1}^{4} \eta_{ij}(\mu; k) t_j$, $i = 1, 2$, $[\eta_{ij}, \eta_{ik}] = 0$, $\det(\eta_{ij}) \neq 0$. Parameter $k$ is complex in general; integration is over the whole complex plane $\Omega_k$ of this parameter. Functions $\Phi_0$, $p_{10}$ and $c_{20}$ are arbitrary; functions $\phi_0$ and $p_0$ solve the following system:

$$\eta_{1j}(\nu; k)\phi_0(\nu, \mu; k) = \eta_{1(j-1)}(\nu; k) \int_{D_{\nu_1}} \phi_0(\nu, \nu_1; k) A(\nu_1, \mu)\, \mathrm{d}\nu_1 \tag{59}$$

$$p_0(\lambda, \nu; k)\eta_{2j}(\nu; k) = \left( \int_{D_{\nu_1}} A(\lambda, \nu_1) p_0(\nu_1, \nu; k)\, \mathrm{d}\nu_1 \right) \eta_{2(j-1)}(\nu; k) \qquad j = 2, 3, \ldots. \tag{60}$$

Functions $\eta_{j1}$ and $\eta_{j2}$ $(j = 1, 2)$ are arbitrary, while $\eta_{jn}$ with $n > 2$ satisfy the following dispersion relations providing compatibility of the equations (59) and (60) with different $j$:

$$\eta^{-1}_{1(j-1)}\eta_{1j} = \eta^{-1}_{1(j-2)}\eta_{1(j-1)} \qquad \eta_{2j}\eta^{-1}_{2(j-1)} = \eta_{2(j-1)}\eta^{-1}_{2(j-2)} \qquad j = 3, 4. \tag{61}$$

We write equation (47) in the explicit form

$$\begin{aligned}\Psi(\lambda, \mu; t) = \delta(\lambda - \mu) &+ \partial^{-1}_{t_1} \left[ \int_{D_{\nu_1}} \int_{D_{\nu_2}} \Phi(\lambda, \nu_1; t) A(\nu_1\nu_2) p_1(\nu_2; t)\, \mathrm{d}\nu_1\, \mathrm{d}\nu_2 \right] p_2(\mu) \\ &+ \partial^{-1}_{t_1} \left[ \int_{D_{\nu_1}} \Phi(\lambda, \nu_1; t) c_1(\nu_1)\, \mathrm{d}\nu_1 B c_2(\mu; t) \right].\end{aligned} \tag{62}$$

Next, we find $U$ from (1): $U = \Psi^{-1} * \Phi$ (we write $\Psi^{-1}_L \equiv \Psi^{-1}$). In general, the operator $\Psi^{-1}$ can be constructed only numerically, unless $\Phi_0$ is degenerate ($\Phi_0(\lambda, \mu; k) = \sum_n \Phi_{1n}(\lambda)\Phi_{2n}(\mu; k)$). In this case $\Psi^{-1}$ may be found analytically, following the procedure proposed, for instance, in [14], where the $\bar{\partial}$-problem with a degenerate kernel has been solved. Of course, this structure of $\Phi_0$ significantly reduces the solution manifold, since the expressions for $h_{nm}$ and $h^j_{nm}$ given by (50)–(52) lose their arbitrary dependence on the variables $t_i$. For instance, this structure may not be used for solving equations with reductions 4 and 5 (section 2, equations (39) and (45)). Of particular interest is the sub-manifold of solutions corresponding to $\Phi_{ij}$ and $c_{20}$ taken in the form $\Phi_{2n}(\mu; k) = \varphi_{2n}(\mu)\delta(k - a_n)$,

$c_{20}(\mu; k) = \sum_n c_{21n}(\mu)\delta(k - b_n)$, where $a_n$ and $b_n$ are (complex) constants. In the limit of $S$-integrable systems, this is a manifold of solitary wave solutions.

Similarly, equations (59) and (60) can be solved numerically, unless $A$ has the following structure: $A(\lambda, \mu) = A_0(\lambda, \mu) + \sum_j A_{j1}(\lambda)A_{j2}(\mu)$, where operator $A_0$ is invertible with a known analytical form for $A_0{}_R^{-1}$. For instance, $A_0(\lambda, \mu) = \delta(\lambda - \mu)$.

Following the above discussion, we can make some points about reductions 4 and 5 of the previous subsection. We have one of equations (40), (44) and (45) relating functions $w_j$ with a given function $F(t)$. This equation fixes the dependence on all parameters $t_i$ in the functions $w_j$. Thus we have to solve equation (46) together with one of these equations and finally find $U(\mu; t)$ and $\Phi_0(\lambda, \nu; k)$. This problem is related to Fredholm-type integral equations with nondegenerate kernels and transcendental algebraic equations. Consequently it can be solved only numerically. But the advantage of these calculations is that the procedure is reduced to operations with spectral parameters instead of operations with independent variables and the associated difficulties are not significantly affected by the dimension $\dim(t)$.

Now we present an example of a simple solution with $N = 3$. Let us take the following solution of the system (11), (13)–(15):

$$\begin{aligned} &B = \alpha\tilde{B} \qquad \tilde{B} = 2\,\mathrm{diag}(1, -1, 2) \qquad A(\lambda, \mu) = A_0(\mu)\delta(\lambda - \mu) \\ &A_0(\mu) = (1 - \mu + \alpha)\tilde{B} \qquad c_2(\mu) = \alpha_2\,\mathrm{e}^{\sum_n B^n t_n} c_{20}(\mu) \\ &\Phi(\lambda, \mu; t) = \Phi_{10}(\lambda)\Phi_{20}(\mu)\,\mathrm{e}^{\sum_n A_0(\mu)^n t_n} \qquad p_0(\mu; t) = \alpha_1\,\mathrm{e}^{\sum_n A_0(\mu)^n t_n} p_{00}(\mu)\delta(\mu - b). \end{aligned} \tag{63}$$

Arbitrary functions of the spectral parameter in these expressions are taken such that each of the matrices $u, q_0, v$ and $w_0$ has only two nonzero elements: $\Phi_{10}(\lambda) = \lambda$, $\Phi_{20}(\mu) = I$, $p_2(\mu) = \alpha_1\delta(\mu - 1)\,\mathrm{diag}(0, 1, 1)$, $c_{20}(\mu) = \delta(\mu - 1)\,\mathrm{diag}(1, 1, 0)$,

$$p_{00}(\mu) = \delta(\mu - 2)\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad c_1(\lambda) = \alpha_2\delta(\lambda - 1)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Thus the nonzero elements of matrices $u, q_0, v, w_0$ are the following:

$$\begin{aligned} &u_{31} = \frac{6\alpha_1\alpha_2}{6\,\mathrm{e}^{\eta_1} + \alpha_1^2\alpha_2^2\,\mathrm{e}^{\eta_2}} && u_{33} = -\frac{2\alpha_1\alpha_2^2}{6\,\mathrm{e}^{\eta_3} + \alpha_1^2\alpha_2^2\,\mathrm{e}^{\eta_3+\eta_2-\eta_1}} \\ &q_{031} = \frac{6\alpha_1^2}{6\,\mathrm{e}^{\eta_4} + \alpha_1^2\alpha_2^2\,\mathrm{e}^{\eta_4+\eta_2-\eta_1}} && q_{033} = \frac{2\alpha_1^2\alpha_2^2}{6\,\mathrm{e}^{\eta_1-\eta_2} + \alpha_1^2\alpha_2^2} \\ &v_{11} = \frac{3\alpha_1^2\alpha_2^2}{6\,\mathrm{e}^{\eta_1-\eta_2} + \alpha_1^2\alpha_2^2} && v_{13} = \frac{6\alpha_2^2}{6\,\mathrm{e}^{\eta_3-\eta_1} + \alpha_1^2\alpha_2^2\,\mathrm{e}^{\eta_3+\eta_2-2\eta_1}} \\ &w_{011} = \frac{3\alpha_1^3\alpha_2}{6\,\mathrm{e}^{\eta_4-\eta_2} + \alpha_1^2\alpha_2^2\,\mathrm{e}^{\eta_4-\eta_1}} && w_{013} = -\frac{6\alpha_1\alpha_2}{6\,\mathrm{e}^{-\eta_2} + \alpha_1^2\alpha_2^2\,\mathrm{e}^{-\eta_1}} \end{aligned}$$

$$\begin{aligned} \eta_1 &= -4\alpha(t_1 + 4\alpha(t_2 + 4\alpha(t_3 + 4\alpha t_4))) \\ \eta_2 &= 2((3\alpha - 2)t_1 + 2((2 - 4\alpha + 3\alpha^2)t_2 + 2((-2 + 6\alpha - 6\alpha^2 + 3\alpha^3)t_3 \\ &\qquad + 2(2 - 8\alpha + 12\alpha^2 - 8\alpha^3 + 3\alpha^4)t_4))) \\ \eta_3 &= -8\alpha(t_1 + \alpha(3t_2 + 2\alpha(5t_3 + 18\alpha t_4))) \\ \eta_4 &= -8(\alpha - 1)(t_1 + 4(\alpha - 1)(t_2 + 4(\alpha - 1)(t_3 + 4(\alpha - 1)t_4))). \end{aligned}$$

Parameters $\alpha_1$ and $\alpha_2$ mark $C$- and $S$-integrable parts of the solution respectively. We see that $q_{033}$ and $v_{11}$ are solitons, while all other functions are unbounded in some direction. Thus reduction 3 (see section 2, equation (34); in our example it corresponds to $\alpha = \alpha_2 = \varepsilon \ll 1$ and $\alpha_1 = 1$) may not be applied to this solution over the whole $(3+1)$-dimensional space; but it may be applied in those regions where $v_{\alpha\beta} \sim \varepsilon^2$, $u_{\alpha\beta} \sim w_{0\alpha\beta} \sim \varepsilon$ and $q_{0\alpha\beta} \sim 1$.

## 3. On the $n$-wave interaction in $(3+1)$ dimensions

In this section we consider equations (7) and (15) with $\chi \neq 0$ and $p_1 = p_2 = 0$. Since $\Phi$ appears only as $\Phi * c_1$, it is convenient to apply operator $*c_1$ to equations (1) from the right. We write the starting system of equations in the form

$$\tilde{\Phi} + \tilde{\chi} = \Psi * \tilde{U} \qquad \tilde{U} = U * c_1 \qquad \tilde{\chi} = \chi * c_1 \tag{64}$$

$$\Psi_{t_j} = \tilde{\Phi} B_j c_2 \tag{65}$$

$$\partial_{t_j} \tilde{\Phi} = \partial_{t_1} \tilde{\Phi} B_j \qquad B_1 = I \tag{66}$$

$$c_{2x_j} = B_j c_{2x_1} \tag{67}$$

where $I$ is the identity matrix, $B_i$ are diagonal matrices. Let

$$\tilde{\chi}_{t_1}(\lambda; t) B_j - \tilde{\chi}_{t_j}(\lambda; t) = \tilde{\chi}(\lambda; t) b_j \tag{68}$$

where $b_j$ are constant $N \times N$ diagonal matrices. We will need the following notation: $V_0 = c_2 * \tilde{U}$ and $V_1 = c_{2t_1} * \tilde{U}$. The nonlinear equation (16) takes the following form after applying operator $*c_1$ to it from the right:

$$\Psi * \left(\partial_{t_j} \tilde{U} - \partial_{t_1} \tilde{U} B_j + \tilde{U}[B_j, V_0]\right) + \tilde{\chi}(b_j - [B_j, V_0]) = 0. \tag{69}$$

Now assume that $\det(b_j - [B_j, V_0]) \neq 0$ for all $j$ and use the two equations (69) with indices $j$ and $k$, $j \neq k$, to eliminate the function $\tilde{\chi}$. Applying operator $c_2 * \Psi^{-1} *$ to the resulting equation from the left we obtain

$$\begin{aligned}\left(\partial_{t_k} V_0 - \partial_{t_1} V_0 B_k + [V_1, B_k] + V_0[B_k, V_0]\right)(b_k - [B_k, V_0])^{-1} \\ = \left(\partial_{t_j} V_0 - \partial_{t_1} V_0 B_j + [V_1, B_j] + V_0[B_j, V_0]\right)(b_j - [B_j, V_0])^{-1}.\end{aligned} \tag{70}$$

To clarify the structure of this equation we note that this is a combination of equations

$$\partial_{t_k} V_0 - \partial_{t_1} V_0 B_k + [V_1, B_k] + V_0[B_k, V_0] = 0 \qquad k > 1$$

which is a hierarchy of $(2+1)$-dimensional matrix $n$-wave equations for off-diagonal elements of $V_0$ (see, for instance, [21]):

$$\partial_{t_k}[B_2, V_0] - \partial_{t_2}[B_k, V_0] - B_2 \partial_{t_1} V_0 B_k + B_k \partial_{t_1} V_0 B_2 + [[B_2, V_0], [B_k, V_0]] = 0 \qquad k > 2.$$

But our system cannot be separated into a set of equations from this list because of the term $\tilde{\chi}$ in equation (64).

Next, let us introduce different scales for the variables $t_k, V_0, V_1$: $\partial_{t_k} \to \varepsilon \partial_{t_k}$, $V_0 = \varepsilon v$, $V_1 = \varepsilon^2 v_1$. Keeping only leading terms (which are of order $\varepsilon^2$), we get from equation (70)

$$\begin{aligned}E_k \equiv v_{t_1}\left(B_j b_j^{-1} - B_k b_k^{-1}\right) + v_{t_k} b_k^{-1} - v_{t_j} b_j^{-1} \\ + [v_1, B_k] b_k^{-1} - [v_1, B_j] b_j^{-1} - v[v, B_j] b_j^{-1} + v[v, B_k] b_k^{-1} = 0.\end{aligned} \tag{71}$$

Combining equations $E_k = 0$ and $E_n = 0$, $k \neq n$, one can eliminate the function $v_1$ and write down an equation for the off-diagonal elements of $v$:

$$E_k\left(B_n b_n^{-1} - B_j b_j^{-1}\right) - E_n\left(B_k b_k^{-1} - B_j b_j^{-1}\right) + B_j(E_k - E_n) b_j^{-1} - B_n E_k b_n^{-1} B_k E_n b_k^{-1} = 0. \tag{72}$$

Let $j = 2, k = 3, n = 4$, $\partial_{t_j} \to \mathrm{i}\partial_{t_j}$; $B_j$ and $b_j$ are real matrices. We write the corresponding equation in the following form:

$$\sum_{\tau=1}^{4} s_{\tau\alpha\beta} \partial_{t_\tau} v_{\alpha\beta} - \mathrm{i} \sum_{\gamma:\gamma\neq\alpha\neq\beta} T_{\alpha\gamma\beta} v_{\alpha\gamma} v_{\gamma\beta} = 0 \qquad \alpha \neq \beta \tag{73}$$

where $s_{\tau\alpha\beta}$ and $T_{\alpha\gamma\beta}$ are constants, expressed in terms of the elements of the matrices $B_j$ and $P_j = (b_k b_n)^{-1}$:

$$s_{k\alpha\beta} = [(B_n)_\beta - (B_n)_\alpha](P_j)_\beta - [(B_j)_\beta - (B_j)_\alpha](P_n)_\beta \qquad k > 1 \tag{74}$$

$$s_{1\alpha\beta} = -\sum_{i=2}^{4}(B_i)_\alpha s_{i\alpha\beta} \tag{75}$$

$$T_{\alpha\gamma\beta} = \sum_{i=2}^{4}[(B_i)_\beta - (B_i)_\gamma]s_{i\alpha\beta}. \tag{76}$$

Thus we have $2 \times 3 \times N$ arbitrary parameters which are elements of the diagonal matrices $B_j$ and $P_j$ ($N \geqslant 3$; otherwise nonlinearity disappears from the system).

Since the fields $v_{\alpha\beta}$ are complex in general, one can conjugate equation (73) and the system of equations for conjugated fields $\bar{v}_{\alpha\beta}$. Because of this, it may be convenient to introduce different fields $w_{\alpha\beta}$ via the formulae

$$v_{\alpha\beta} = w_{\alpha\beta} \quad \beta > \alpha \qquad v_{\alpha\beta} = \bar{w}_{\alpha\beta} \quad \beta < \alpha \tag{77}$$

(these equations do not establish any relation among the functions $v_{\alpha\beta}$!) and write equation (73) in terms of these fields:

$$\sum_{\tau=1}^{4} s_{\tau\alpha\beta}\partial_{t_\tau} w_{\alpha\beta} - \mathrm{i}\sum_{\gamma:\alpha<\gamma<\beta} T_{\alpha\gamma\beta} w_{\alpha\gamma} w_{\gamma\beta} - \mathrm{i}\sum_{\gamma:\alpha<\beta<\gamma} T_{\alpha\gamma\beta} w_{\alpha\gamma}\bar{w}_{\gamma\beta}$$
$$- \mathrm{i}\sum_{\gamma:\gamma<\alpha<\beta} T_{\alpha\gamma\beta}\bar{w}_{\alpha\gamma} w_{\gamma\beta} = 0 \qquad \alpha < \beta \tag{78}$$

$$\sum_{\tau=1}^{4} s_{\tau\alpha\beta}\partial_{t_\tau} \bar{w}_{\alpha\beta} - \mathrm{i}\sum_{\gamma:\alpha>\gamma>\beta} T_{\alpha\gamma\beta} \bar{w}_{\alpha\gamma} \bar{w}_{\gamma\beta} - \mathrm{i}\sum_{\gamma:\gamma>\alpha>\beta} T_{\alpha\gamma\beta} w_{\alpha\gamma}\bar{w}_{\gamma\beta}$$
$$- \mathrm{i}\sum_{\gamma:\gamma<\beta<\alpha} T_{\alpha\gamma\beta}\bar{w}_{\alpha\gamma} w_{\gamma\beta} = 0 \qquad \alpha > \beta \tag{79}$$

Conjugate the last equation:

$$\sum_{\tau=1}^{4} s_{\tau\alpha\beta}\partial_{t_\tau} w_{\alpha\beta} + \mathrm{i}\sum_{\gamma:\alpha>\gamma>\beta} T_{\alpha\gamma\beta} w_{\alpha\gamma} w_{\gamma\beta} + \mathrm{i}\sum_{\gamma:\gamma>\alpha>\beta} T_{\alpha\gamma\beta} \bar{w}_{\alpha\gamma} w_{\gamma\beta}$$
$$+ \mathrm{i}\sum_{\gamma:\gamma<\beta<\alpha} T_{\alpha\gamma\beta} w_{\alpha\gamma}\bar{w}_{\gamma\beta} = 0 \qquad \alpha > \beta \tag{80}$$

Thus the general system of nonlinear PDE is formed by equations (78) and (80).

Now we consider several reductions with the following abbreviated form for the linear differential operator:

$$L_{\alpha\beta} = \sum_{\tau=1}^{4} s_{\tau\alpha\beta}\partial_{t_\tau}. \tag{81}$$

1. *Reduction to the classical* (2 + 1)-*dimensional n-wave equation*. Let $P_n = B_n$, which reduces the number of arbitrary parameters to $3 \times N$. Then for coefficients

of equation (73) we get

$$s_{k\alpha\beta} = (B_j)_\alpha(B_n)_\beta - (B_n)_\alpha(B_j)_\beta \qquad s_{1\alpha\beta} = 0 \qquad T_{\alpha\gamma\beta} = \sum_{i=2}^{4}[(B_i)_\beta - (B_i)_\gamma]s_{i\alpha\beta} \tag{82}$$

i.e. equation (73) becomes the usual $S$-integrable (2+1)-dimensional $n$-wave system [21].

2. *Reduction to the system with real fields.* Let $v_{\alpha\beta} = \mathrm{i}w_{\alpha\beta}$, where $w_{\alpha\beta}$ are real functions. Equation (73) reads

$$\sum_{\tau=1}^{4} s_{\tau\alpha\beta}\partial_{t_\tau} w_{\alpha\beta} + \sum_{\gamma:\gamma\neq\alpha\neq\beta} T_{\alpha\gamma\beta} w_{\alpha\gamma} w_{\gamma\beta} = 0 \qquad \alpha \neq \beta. \tag{83}$$

3. *The particular example of six-wave interaction.* Consider the $n$-wave system composed of equations (78) and (80). Let $N = 3$; equations (78) and (80) result in a system of six equations:

$$L_{12}w_{12} = \mathrm{i}T_{132}w_{13}\bar{w}_{32} \qquad L_{13}w_{13} = \mathrm{i}T_{123}w_{12}w_{23} \qquad L_{23}w_{23} = \mathrm{i}T_{213}\bar{w}_{21}w_{13} \tag{84}$$

$$L_{21}w_{21} = -\mathrm{i}T_{231}\bar{w}_{23}w_{31} \qquad L_{31}w_{31} = -\mathrm{i}T_{321}w_{32}w_{21} \qquad L_{32}w_{32} = -\mathrm{i}T_{312}w_{31}\bar{w}_{12}. \tag{85}$$

Thus equations (84) and (85) compose a complete system describing the particular case of six-wave interaction. Note that the formulae for $w_{21}$, $w_{31}$ and $w_{32}$ are known only in terms of their complex conjugate values due to the definitions (77).

### 3.1. Construction of solutions

In general, this section is similar to section 2.1, but has different features.

The general solution to the nonlinear system can be represented in the form

$$\tilde{U}(t) = \tilde{\Phi}(t) + \tilde{\chi}(t) - \partial_{t_1}^{-1}[\tilde{\Phi}(t)c_2(t)] * \tilde{U}(t) \tag{86}$$

where $\tilde{\Phi}$, $c_2$ and $\tilde{\chi}$ are solutions of equations (66), (67) and (68) respectively. Thus these functions have an arbitrary dependence on a single variable and the field $V_0 = c_2 * \tilde{U}$ has an arbitrary dependence on two variables at most, due to the terms

$$c_2 * \Phi \qquad c_2 * \chi \qquad c_2 * \partial_{t_1}^{-1}[\Phi * c_1c_2] \tag{87}$$

(in accordance with section 2.1).

Below we give the algorithm for construction of the function $V_0$, which is a solution of the equation (70). Let us take solutions of equations (66) and (68) in the form

$$\tilde{\Phi}(\lambda; t) = \int_{\Omega_k} \tilde{\Phi}_0(\lambda, k)\,\mathrm{e}^{k\sum_n B_n t_n}\,\mathrm{d}k \tag{88}$$

$$c_2(\lambda; t) = \int_{\Omega_k} \mathrm{e}^{k\sum_n B_n t_n} c_{20}(\lambda, k)\,\mathrm{d}k \tag{89}$$

$$\tilde{\chi}(\lambda; t) = \int_{\Omega_k} \tilde{\chi}_0(\lambda; k)\,\mathrm{e}^{\sum_n A_n(k)t_n}\,\mathrm{d}k \qquad A_1(k)B_j - A_j(k) = b_j. \tag{90}$$

We integrate equation (65) to find $\Psi$ ($j = 1$; remember that $B_1 = I$):

$$\Psi(\lambda, \mu) = \int_{\Omega_k}\int_{\Omega_q} \tilde{\Phi}_0(\lambda; k)\,\mathrm{e}^{(k+q)\sum_n B_n t_n} c_{20}(\mu, q)\frac{\mathrm{d}k\,\mathrm{d}q}{k+q} + \delta(\lambda - \mu). \tag{91}$$

Thus

$$\tilde{U}(\lambda) = \tilde{\Phi}(\lambda) - \int_{\Omega_k}\int_{\Omega_q} \tilde{\Phi}_0(\lambda; k)\, \mathrm{e}^{(k+q)\sum_n B_n t_n} \phi(q) \frac{\mathrm{d}k\, \mathrm{d}q}{k+q} + \tilde{\chi}(\lambda) \qquad \phi = c_{20} * \tilde{U} \tag{92}$$

and

$$V_0 = c_2 * \tilde{U}. \tag{93}$$

The unknown function $\phi$, related to $\tilde{U}$, can be found only numerically in the general case, unless the functions $\tilde{\Phi}(\lambda, k)$ are degenerate [14]: $\tilde{\Phi}(\lambda, k) = \sum_n \Phi_{1n}(\lambda)\Phi_{2n}(k)$. Of course, this structure destroys the arbitrary dependence on two variables in expressions (87). In particular, $\tilde{\Phi}(\lambda, k) = \sum_n \varphi_n(\lambda)\delta(k - a_n)$, $c_{20}(\lambda; k) = \sum_n c_{2n}(\lambda)\delta(k - b_n)$ with (complex) constants $a_k, b_k$ gives solitary wave solutions in the limit of classical $S$-integrable systems.

We present a simple example of a solution. Let $N = 3$, $\lambda$, $\mu$ and $\nu$ be real parameters, $D_\lambda = D_\mu = D_\nu = (-\infty, \infty)$, $B_2 = \mathrm{diag}(1, 1, -1)$, $B_3 = \mathrm{diag}(1, -1, -1)$, $B_4 = \mathrm{diag}(-1, -1, -1)$, $\tilde{\chi}(\lambda) = \chi_0(\lambda)\, \mathrm{e}^{\sum_n r_n t_n}$ (thus $b_j = r_1 B_j - r_j$), $r_1 = B_1^2 = I, r_j = 2B_j^2$, $j > 1$, $c_1(\lambda) = \lambda$, $c_{20}(\lambda; q) = \mathrm{diag}(\mathrm{e}^{-4\lambda^2\pi}, \mathrm{e}^{-\lambda^2\pi}, \mathrm{e}^{-4\lambda^2\pi})\delta(q - 1)$, $\Phi_0(\lambda, \mu; k) = \Phi_{01}(\lambda)\Phi_{02}(\mu)\delta(k-1)$, $\Phi_{02}(\mu) = \mu\, \mathrm{diag}(\mathrm{e}^{-9\mu^2\pi}, \mathrm{e}^{-\mu^2\pi}, \mathrm{e}^{-9\mu^2\pi})$ and $\Phi_{01}(\lambda) = \chi_0(\lambda) = \alpha_1 I_I$, where $I_I$ is a $3 \times 3$ matrix of units (i.e. independent of $\lambda$). Then $V_0$ can be found explicitly:

$$V_0 = \mathrm{d}(t) \times$$
$$\begin{pmatrix} 2\,\mathrm{e}^{4(t_2+t_3)}(1 + 54\,\mathrm{e}^{t_2+t_3+3t_4}\pi) & 54\,\mathrm{e}^{2(2t_2+t_3)}(1 + 2\,\mathrm{e}^{t_2+3(t_3+t_4)}\pi) & 2\,\mathrm{e}^{2(t_2+t_3)}(1 + 54\,\mathrm{e}^{3(t_2+t_3+t_4)}\pi) \\ 4\,\mathrm{e}^{2(2t_2+t_3)}(1 + 54\,\mathrm{e}^{t_2+t_3+3t_4}\pi) & 108\,\mathrm{e}^{4t_2}(1 + 2\,\mathrm{e}^{t_2+3(t_3+t_4)}\pi) & 4\,\mathrm{e}^{2t_2}(1 + 54\,\mathrm{e}^{3(t_2+t_3+t_4)}\pi) \\ 2\,\mathrm{e}^{2(t_2+t_3)}(1 + 54\,\mathrm{e}^{t_2+t_3+3t_4}\pi) & 54\,\mathrm{e}^{2t_2}(1 + 2\,\mathrm{e}^{t_2+3(t_3+t_4)}\pi) & 2(1 + 54\,\mathrm{e}^{3(t_2+t_3+t_4)}\pi) \end{pmatrix}$$
$$\mathrm{d}(t) = \alpha_1/(\alpha_1 + 54\alpha_1\, \mathrm{e}^{4t_2} + \alpha_1\, \mathrm{e}^{4(t_2+t_3)} + 216\, \mathrm{e}^{2(-t_1+t_2+t_3+t_4)}).$$

As regards the multiple-scale expansion of equation (70), given by equations (71), one should make the replacement $t_j \to \varepsilon t_j$ in formulae (88)–(93) and take $\alpha_1 = \varepsilon$. Thus $V_0 \sim \epsilon$. In addition, one needs to introduce an imaginary unit, $t_k \to i t_k$, to produce a solution of equation (73) and guarantee finiteness of $V_0$ in the $(3 + 1)$-dimensional space of the parameter $t$ (or equations (84) and (85) since $N = 3$).

## 4. Conclusions

Working with dressing methods we underline two directions: (a) increase of the dimension of solvable nonlinear PDE and (b) providing a rich class of solutions of them. The nonlinear PDE derived with our algorithm allow an infinite set of commuting flows corresponding to different parameters $t_j$. Since general equations are rather complicated (see equations (24), (28) and (70)), a reasonable problem is the construction of reductions of them that would show physical applications of these systems (see reductions 1, 2, 4 in section 2). Another approach is multiple-scale expansion of the general systems (see reductions 3, 5 in section 2 and section 3). For instance, this reveals a $(3 + 1)$-dimensional $n$-wave system (see equation (73)).

We should emphasize the importance of reductions 4 and 5 in section 2 (see equations (39) and (45)), which may be used for any $S$- or $C$-integrable system. The possibility of introducing an arbitrary forcing term is due to the presence of both $S$- and $C$-integrable parts in formula (8), since the fields $w_j$ join functions coming from $S$- and $C$-integrable parts. Taking the forcing term as a function of the fields ($v$ in the equation (40) or $q$ in equation (45)), one enriches the manifold of solvable equations, although this significantly complicates the solution construction.

Our first example combining $S$- and $C$-integrable systems seems to have enough freedom for providing a full description of a solution manifold. One should to use this freedom properly, which is not simple problem. The second example (section 3) contrasts with the first one. Remember that our algorithm supplies solutions with an arbitrary dependence on two independent variables at most in this case. Obviously, this is not enough for providing a full description of a $(3 + 1)$-dimensional solution manifold.

## Acknowledgments

The author thanks Professors S V Manakov and P M Santini for discussions and a referee for useful comments. This work was supported by RFBR grants 03-01-06122 and 1716.2003.1.

## References

[1] Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1984 *Theory of Solitons. The Inverse Problem Method* (New York: Plenum)
[2] Ablowitz M J and Segur H 1981 *Solitons and Inverse Scattering Transform* (Philadelphia: SIAM)
[3] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[4] Calogero F 1990 *What is Integrability* ed V E Zakharov (Berlin: Springer) p 1
[5] Calogero F and Xiaoda Ji 1991 *J. Math. Phys.* **32** 875
[6] Calogero F and Xiaoda Ji 1991 *J. Math. Phys.* **32** 2703
[7] Calogero F 1992 *J. Math. Phys.* **33** 1257
[8] Calogero F 1993 *J. Math. Phys.* **34** 3197
[9] Calogero F and Xiaoda Ji 1993 *J. Math. Phys.* **34** 5810
[10] Zakharov V E and Shabat A B 1974 *Funct. Anal. Appl.* **8** 43
[11] Zakharov V E and Shabat A B 1979 *Funct. Anal. Appl.* **13** 13
[12] Zakharov V E and Manakov S V 1985 *Funct. Anal. Appl.* **19** 11
[13] Bogdanov L V and Manakov S V 1988 *J. Phys. A: Math. Gen.* **21** L537
[14] Konopelchenko B 1993 *Solitons in Multidimensions* (Singapore: World Scientific)
[15] Date E, Kashiwara M, Jimbo M and Miwa T 1983 *Nonlinear integrable Systems—Classical Theory and Quantum Theory* ed M Jimbo and T Miwa (Singapore: World Scientific) p 39
[16] Jimbo M and Miwa T 1983 *Publ. RIMS, Kyoto Univ.* **19** 943
[17] Sato M and Sato Y 1983 *Nonlinear Partial Differential Equations in Applied Science* ed H Fujita, P D Lax and G Strang (Tokyo: Kinokuniya/North-Holland) p 259
[18] Ohta Y, Satsuma J, Takahashi D and Tokihiro T 1988 *Prog. Theor. Phys. Suppl.* **94** 210
[19] Zenchuk A I 2003 *JETP Lett.* **77** 376
[20] Kaup D J, Rieman A and Bers A 1979 *Rev. Mod. Phys.* **51** 275
[21] Zakharov V E 1998 *Duke Math. J.* **94** 103–139